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Minimal orbits of metrics

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Abstract

The group of diffeomorphisms of a compact manifold acts isometrically on the space of Riemannian metrics with its L^2 metric. Following Arnaudon and Paycha (1995) and Maeda, Rosenberg and Tondeur (1993), we define minimal orbits for this action by a zeta function regularization. We show that odd dimensional isotropy irreducible homogeneous spaces give rise to minimal orbits, the first known examples of minimal submanifolds of infinite dimension and codimension. We also find a flat 2-torus giving a stable minimal orbit. We prove that isolated orbits are minimal, as in finite dimensions.

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1. Introduction

Let X be a Riemannian manifold with an isometric action of a Lie group. If X is finite dimensional, it follows from Hsiang's theorem [7] that orbits of minimal volume among all nearby orbits of the same type are in fact minimal submanifolds of X. Gauge theory provides an infinite dimensional analogue of this situation, where X is the space of connections on

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a principal bundle over a compact manifold M, and the Lie group is the gauge group. In [8,14] minimal orbits were defined in this context by a zeta function regularization and examples of minimal orbits were given. Zeta functions enter the discussion since in finite dimensions the first variation formula computes the variation of the determinant of the metric on a submanifold; in infinite dimensions this determinant is formally the determinant of a Laplacian-type operator and is defined by Ray–Singer/zeta function regularization. Thus the infinite dimensional geometry of the space of connections is related to attempts to quantize Yang–Mills theory, since regularized determinants are also a key element of semiclassical Yang–Mills theory.

The regularization actually computes $\mathrm{Tr}_N\,\mathrm{II}$, the component of the trace of the second fundamental form in the direction N; an orbit is minimal if $\mathrm{Tr}_N\,\mathrm{II} = 0$ for all N. The regularization in [14] had the disadvantage of being finite only for certain orbits. In [1], a term was added to the regularization which guarantees that the new regularized definition of $\mathrm{Tr}_N\,\mathrm{II}$ is always finite. This counterterm is zero in finite dimensions, so both regularizations generalize the usual notion of $\mathrm{Tr}_N\,\mathrm{II}$.

In this paper we treat a different physically interesting case of an infinite dimensional Riemannian manifold with an isometric action of an infinite dimensional group. Here the manifold is $\mathcal{M}$, the space of Riemannian metrics on a fixed compact manifold X, and the group $\mathcal{D}$ is the space of diffeomorphisms of X. The determinants that appear here should relate ultimately to quantum gravity.

It turns out that the case of Riemannian metrics is technically more difficult to handle than the gauge theory case, as here the group carries no natural metric. The resulting Laplacians used to define the regularization thus depend on a fixed choice of metric on X, and these non-natural Laplacians must be related to the natural Laplacians that have appeared previously in discussions of $\mathcal{M}$. The theory also becomes more complicated when orbits of varying type occur. The main results are as follows (Theorems 3.1–3.3 and 3.5).

Theorem 1.1.

(i) *In odd dimensions, the orbit of the volume one G-invariant metric on an isotropy irreducible homogeneous space G/H is minimal within the space of volume one metrics on G/H.*

(ii) *The orbits of the flat 2-tori of volume one associated to the the points* $(0, 1)$ *and* $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ *in the upper half plane are minimal within the space of all flat tori of volume one. The orbit associated to* $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ *is a stable minimal orbit.*

(iii) *An orbit of isolated diffeomorphism type is minimal.*

Note that the isotropy irreducible homogeneous spaces include the symmetric spaces, but many more examples exist. These spaces are minimal orbits of infinite dimension and codimension. Part (iii) is an easy corollary of Hsiang's theorem in finite dimensions, but is non-trivial in infinite dimensions.

The paper is organized as follows. In Section 2 the first variation formula in finite dimensions is rederived in terms of the zeta function of a finite dimensional transformation which is the analogue of the Laplacians appearing in infinite dimensions. This serves as motivation for the later sections. We also discuss the effect of varying the metric on the submanifold, which is an unnecessary complication in finite dimensions but is forced upon us in infinite dimensions.

In Section 3 we handle the case of $\mathcal{M}$. Section 3.1 gives the general theory in infinite dimensions, Section 3.2 computes minimal orbits of flat 2-tori, Section 3.3 treats the case of orbits of varying type and discusses isotropy irreducible homogeneous spaces, and Section 3.4 gathers some local computations of the Laplacians used.

2. The finite dimensional case

In this section we rederive the first variation formula for an immersed submanifold M of a Riemannian manifold $(\overline{M}, \overline{g})$ in terms of the eigenvalues of a finite dimensional operator. Each step in this calculation has counterparts in the usual derivation (cf. [12, Ch. 1]). Afterwards, we modify the first variation formula to motivate some calculations in infinite dimensions.

Let $i : M \to \overline{M}$ be the immersion and set $L = L_x = \mathrm{d}i_x : T_xM \to T_{i(x)}\overline{M}$. We fix a Riemannian metric g on M. To be consistent with the notation in the rest of the paper, we set $\overline{\Delta} = \overline{\Delta}_x = L^*L : T_xM \to T_xM$. (To be strictly consistent, we should relabel L^* as $\overline{L}^*$.) There exists an orthonormal basis $\{\phi_i\}$ of T_xM consisting of eigenfunctions of $\overline{\Delta}$, i.e. $\overline{\Delta}\phi_i = \lambda_i\phi_i$. Since i is an immersion, $\lambda_i > 0$. If we let $X_i = L\phi_i/\sqrt{\lambda_i}$, then $\{X_i\}$ is an orthonormal basis of $T_{i(x)}i(M)$. For a fixed $x \in M$, we may extend $\{\phi_i\}$ near x so that $\overline{\Delta}_y\phi_i(y) = \lambda_i(y)\phi_i(y)$ for all y in the neighbourhood U of x.

Take a variation $F : M \times (-\epsilon, \epsilon) \to \overline{M}$ with variation vector field $N_x = \mathrm{d}F_{(x,0)}(\partial_\alpha)$, where α is the parameter for $(-\epsilon, \epsilon)$. We assume $N \perp i(M)$. (Strictly speaking, $N \in \Gamma(i^*T\overline{M})$, but near x we may write $N \in \Gamma(T\overline{M})$.) Let X^ν denote the projection of a vector $X \in T\overline{M}$ into the normal bundle to $\mathrm{d}i(TM)$ (which is locally defined) in $T\overline{M}$. Then $\mathrm{Tr}_N\,\mathrm{II}$, the component of the trace of the second fundamental form at x in the direction of $N = N_x$, is by definition

$$\mathrm{Tr}_N\,\mathrm{II} = \langle(\nabla_{X_i}X_i)^\nu, N\rangle_{\overline{g}},$$

where ∇ is the Levi-Civita connection on $\overline{M}$ and we are using summation convention. Here we omit mentioning the point x in $\mathrm{Tr}_N\,\mathrm{II}$. Using $\langle X, N\rangle = 0$, we get

$$\begin{aligned}\mathrm{Tr}_N\,\mathrm{II} &= \langle\nabla_{X_i}X_i, N\rangle = -\langle X_i, \nabla_{X_i}N\rangle \\ &= -\left\langle\frac{L\phi_i}{\sqrt{\lambda_i}}, \nabla_{L\phi_i/\sqrt{\lambda_i}}N\right\rangle = -\frac{1}{\lambda_i}\langle L\phi_i, \nabla_{L\phi_i}N\rangle.\end{aligned}$$

Here and from now on all inner products are with respect to $\bar{g}$ unless otherwise noted. We now extend ϕ_i, N to vector fields on $U \times (-\epsilon, \epsilon)$, $F(U \times (-\epsilon, \epsilon))$, respectively, by trivially setting $\phi_i(y, \alpha) \stackrel{\mathrm{def}}{=} \phi_i(y, 0)$ and setting $N_{F(x,\alpha)} = \mathrm{d}F_{(x,\alpha)}(\partial_\alpha)$. L also extends to the operator $\mathrm{d}F : TU \times (-\epsilon, \epsilon) \to T\overline{M}$. Thus

$$\mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{\lambda_i}\langle L\phi_i, \nabla_N L\phi_i\rangle - \frac{1}{\lambda_i}\langle L\phi_i, [L\phi_i, N]\rangle.$$

The last term vanishes, since $[L\phi_i, N] = [\mathrm{d}F(\phi_i), \mathrm{d}F(\partial_\alpha)] = \mathrm{d}F[\phi_i, \partial_\alpha] = \mathrm{d}F(0) = 0$, and so

$$\mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{\lambda_i}\langle L\phi_i, \nabla_N L\phi_i\rangle. \tag{2.1}$$

Remark. Let $G = \overline{\Delta}^{-1}$ be the "Green's operator" for $\overline{\Delta}$. Then (2.1) becomes

$$\begin{aligned}\mathrm{Tr}_N\,\mathrm{II} &= -\langle LG\phi_i, \nabla_N L\phi_i\rangle = -\langle \phi_i,\ GL^*\nabla_N L\phi_i\rangle \\ &= -\mathrm{Tr}(GL^*\nabla_N L) \\ &= -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}(\mathrm{e}^{-t\overline{\Delta}}L^*\nabla_N L)\,\mathrm{d}t|_{s=1} \\ &= \int_0^\infty t^s\,\mathrm{Tr}(\mathrm{e}^{-t\overline{\Delta}}L^*\nabla_N L)\,\mathrm{d}t|_{s=0}. \end{aligned} \tag{2.2}$$

Let $\overline{\zeta}(s) = \sum_i(\lambda_i)^{-s}$ be the zeta function of $\overline{\Delta}$. Then $\overline{\zeta}(0) = \dim M$, and so the variation of $\overline{\zeta}(0)$ in the direction N satisfies $\delta_N\overline{\zeta}(0) = 0$. Thus we may rewrite $\mathrm{Tr}_N\,\mathrm{II}$ as

$$\mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}(\mathrm{e}^{-t\overline{\Delta}}L^*\nabla_N L)\,\mathrm{d}t|_{s=1} - \left.\frac{\delta_N\overline{\zeta}(0)}{2(s-1)}\right|_{s=1}. \tag{2.3}$$

Following [1,14], we will use (2.3) as the regularization of $\mathrm{Tr}_N\,\mathrm{II}$ in infinite dimensions. (In [14], the next to last line in (2.2) was used as the regularization, and the importance of the last term was shown in [1].)

Continuing with the derivation of the first variation formula, we set $\overline{g}_\alpha$ to be the restriction of $\overline{g}$ to $T_{F(x,\alpha)}F(M\times\{\alpha\})$, and let

$$L_\alpha = \mathrm{d}F_{(x,\alpha)}|_{TM\times\alpha}.$$

By (2.1), we have

$$\begin{aligned}\mathrm{Tr}_N\,\mathrm{II} &= -\frac{1}{\lambda_i}\langle L\phi_i, \nabla_N L\phi_i\rangle = -\frac{1}{2\lambda_i}N\langle L\phi_i, L\phi_i\rangle \\ &= -\frac{1}{2\lambda_i}\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0}\langle L_\alpha\phi_i, L_\alpha\phi_i\rangle_{\overline{g}_\alpha} = -\frac{1}{2\lambda_i}\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0}\langle\overline{\Delta}_\alpha\phi_i, \phi_i\rangle_g \\ &= -\frac{1}{2\lambda_i}\langle(\delta_\alpha\overline{\Delta}_\alpha)\phi_i, \phi_i\rangle_g, \end{aligned} \tag{2.4}$$

where $\delta_\alpha\overline{\Delta}$ denotes the variation of $\overline{\Delta}$ in the direction α. This expression is independent of the extension of the orthonormal basis $\{\phi_i\}$ on U to an orthonormal basis on $U\times(-\epsilon,\epsilon)$. So extend $\{\phi_i\}$ to $\{\phi_i(y,\alpha)\}$ on $U\times(-\epsilon,\epsilon)$ so that $\overline{\Delta}_\alpha\phi_i(\alpha) = \lambda_i(\alpha)\phi_i(\alpha)$ (dropping y from the notation). Note that this is not the same $\phi(y,\alpha)$ as before. Then for $\delta = \delta_\alpha$ and $\dot{\lambda}_i = (\mathrm{d}/\mathrm{d}\alpha)|_{\alpha=0}\lambda_i$, the formula $(\delta\overline{\Delta}_\alpha)\phi_i + \overline{\Delta}_\alpha(\delta\phi_i) = \dot{\lambda}_i\phi_i + \lambda_i\delta\phi_i$ at $\alpha = 0$ yields

$$
\begin{aligned}
\langle(\delta\overline{\Delta}_\alpha)\phi_i,\phi_i\rangle_g &= -\langle\delta\phi_i,\overline{\Delta}\phi_i\rangle_g + \langle\dot{\lambda}_i\phi_i,\phi_i\rangle_g + \langle\lambda_i\delta\phi_i,\phi_i\rangle_g \\
&= -\lambda_i\langle\delta\phi_i,\phi_i\rangle_g + \dot{\lambda}_i + \lambda_i\langle\delta\phi_i,\phi_i\rangle_g \\
&= \dot{\lambda}_i.
\end{aligned}
\tag{2.5}
$$

Combining (2.4) and (2.5) gives

$$
\mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{2}\sum_i \frac{\dot{\lambda}_i}{\lambda_i}. \tag{2.6}
$$

We remark that there may be trouble defining $\dot{\lambda}_i$ where an eigenvalue bifurcates, but this difficulty disappears when we sum over i, so the computation above is valid.

For $\overline{\zeta}(s) = \sum_i (\lambda_i)^{-s}$, it is easy to check from (2.6) that

$$
\mathrm{Tr}_N\,\mathrm{II} = \tfrac{1}{2}\delta\overline{\zeta}'(0).
$$

An infinite dimensional analogue is given in (3.9).

Now let $\{\phi_i^*\}$ be the frame of T_x^*M dual to $\{\phi_i\}$, and let dvol be the volume form for $i(M)$ at $i(x)$. Then

$$
\begin{aligned}
-\,\mathrm{Tr}_N\,\mathrm{II}\,\mathrm{dvol} &= \frac{1}{2}\sum_i \frac{\dot{\lambda}_i}{\lambda_i}\det{}^{1/2}(\langle L\phi_i, L\phi_j\rangle)\phi_1^*\wedge\cdots\wedge\phi_n^* \\
&= \frac{1}{2}\sum_i \frac{\dot{\lambda}_i}{\lambda_i}\det{}^{1/2}(\langle\overline{\Delta}\phi_i,\phi_j\rangle)\phi_1^*\wedge\cdots\wedge\phi_n^* \\
&= \frac{1}{2}\sum_i \frac{\dot{\lambda}_i}{\lambda_i}\left(\prod_i \lambda_i\right)^{1/2}\phi_1^*\wedge\cdots\wedge\phi_n^* \\
&= \frac{1}{2}\sum_i \left(\prod_i \lambda_i\right)^{-1/2}(\dot{\lambda}_1\lambda_2\cdots\lambda_n + \cdots + \lambda_1\cdots\lambda_{n-1}\cdot\dot{\lambda}_n) \\
&\qquad\qquad \times\phi_1^*\wedge\cdots\wedge\phi_n^* \\
&= N\left(\left(\prod_i \lambda_i\right)^{1/2}\right)\phi_1^*\wedge\cdots\wedge\phi_n^*.
\end{aligned}
\tag{2.7}
$$

Also, the volume form at $F(x,\alpha)$ is given by

$$
\det{}^{1/2}(\langle L_\alpha\phi_i, L_\alpha\phi_j\rangle)\phi_1^*\wedge\cdots\wedge\phi_n^* = \left(\prod \lambda_i(\alpha)\right)^{1/2}\phi_1^*\wedge\cdots\wedge\phi_n^* \tag{2.8}
$$

(cf. [12, p. 8]). Combining (2.7) and (2.8) gives

$$
-\,\langle\mathrm{Tr}\,\mathrm{II}, N\rangle\,\mathrm{dvol} = -\mathrm{Tr}_N\,\mathrm{II}\,\mathrm{dvol} = N(\mathrm{dvol}). \tag{2.9}
$$

This is the first variation formula, which is usually written in the global form

$$
-\int_M \langle\mathrm{Tr}\,\mathrm{II}, N\rangle\,\mathrm{dvol} = N\left(\int_M \mathrm{dvol}\right).
$$

We now discuss the effect of allowing the metric g_α on $M \times \{\alpha\}$ to vary with α. Of course, there is no need for this complication in finite dimensions, but it cannot be avoided in the next section.

So put the metric $g_\alpha \oplus \mathrm{d}\alpha^2$ on $M \times (-\epsilon, \epsilon)$, where $g_0 = g$. By (2.4), we have

$$\mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{2\lambda_i}\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0}\langle L_\alpha\phi_i, L_\alpha\phi_i\rangle_{\overline{g}_\alpha} = -\frac{1}{2\lambda_i}\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0}\langle \Delta_\alpha\phi_i, \phi_i\rangle_{g_\alpha}.$$

Here $\Delta_\alpha = L_\alpha^* L_\alpha$, where L_α^* is now defined by

$$\langle L_\alpha\phi, \psi\rangle_{\overline{g}_\alpha} = \langle \phi, L_\alpha^*\psi\rangle_{g_\alpha}.$$

Thus

$$\mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{2\lambda_i}\langle(\delta\Delta_\alpha)\phi_i, \phi_i\rangle_g - \frac{1}{2\lambda_i}N_{ab}(\Delta\phi_i)^a\phi_i^b,$$

where $N_{ab} = (\mathrm{d}/\mathrm{d}\alpha)|_{\alpha=0}g_\alpha$. Thus we can write δ_N for $\delta = \delta_\alpha$. Using $\Delta\phi_i = \lambda_i\phi_i = \overline{\Delta}\phi_i$ at $\alpha = 0$, we get

$$\begin{aligned}\mathrm{Tr}_N\,\mathrm{II} &= -\frac{1}{2}\sum_i\langle\delta\Delta_\alpha\cdot\Delta^{-1}\phi_i, \phi_i\rangle_g - \frac{1}{2}\sum_i N_{ab}\phi_i^a\phi_i^b \\ &= -\frac{1}{2\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}_g(\delta_N\Delta_\alpha\cdot \mathrm{e}^{-t\Delta})\,\mathrm{d}t|_{s=1} \\ &\quad -\frac{1}{2}\sum_i g_{bs}g^{sc}N_{ca}\phi_i^a\phi_i^b.\end{aligned}$$

Now set

$$\zeta_N(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}_g(\delta_N\Delta_\alpha\cdot\mathrm{e}^{-t\Delta})\,\mathrm{d}t,$$

$$\overline{\zeta}_N(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}_g(\delta_N\overline{\Delta}_\alpha\cdot\mathrm{e}^{-t\Delta})\,\mathrm{d}t.$$

The calculation above gives

$$\mathrm{Tr}_N\,\mathrm{II} = -\tfrac{1}{2}(\zeta_N(1) + \mathrm{Tr}(\tilde{N})),$$

where $(\tilde{N}\phi)^s = g^{sc}N_{ca}\phi^a$, i.e. $\tilde{N}$ lowers an index by N and raises an index by g. Similarly, repeating the calculation above starting at (2.4) but now using $\overline{\Delta}_\alpha$, we obtain

$$\mathrm{Tr}_N\,\mathrm{II} = -\tfrac{1}{2}\overline{\zeta}_N(1).$$

Thus $\overline{\zeta}_N(1) = \zeta_N(1) + \mathrm{Tr}(\tilde{N})$.

Finally,

$$\mathrm{Tr}(\tilde{N}) = \mathrm{Tr}(\tilde{N}\Delta^{-s})|_{s=0} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}(\tilde{N}\mathrm{e}^{-t\Delta})\,\mathrm{d}t|_{s=0}.$$

Thus

$$\mathrm{Tr}_N\,\mathrm{II} = \overline{\zeta}_N(1) = \zeta_N(1) + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}(\tilde{N}\mathrm{e}^{-t\Delta})\,\mathrm{d}t|_{s=0}.$$

This formula is the finite dimensional analogue of (3.16), with the operator B in (3.16) a slightly more complicated version of $\tilde{N}$ and the added difficulty that $\ker \Delta$ need not vanish. While it seems unnatural in finite dimensions to work with $\zeta_N(1)$, it turns out to be the natural choice in infinite dimensions.

3. Metrics and diffeomorphisms

In this section we will apply the first variation formula of Section 2 to the infinite dimensional situation of the orbits of the diffeomorphism group of a compact manifold M within the space of Riemannian metrics on M. In particular, we will define what it means for an orbit to be minimal within the space of metrics, and relate this minimality to the determinant of a Laplacian-type operator. This is similar to the gauge theory case considered in [8,14], and the general theory in [1], but has extra complications arising from the lack of a natural metric on the gauge group. We also produce several examples of minimal orbits.

In Section 3.1, we set up the general theory when all orbits have the same diffeomorphism type. We will apply this to find minimal orbits of flat 2-tori in Section 3.2. For other examples of minimal orbits, we need to treat the case of orbits of varying type. This is done in Section 3.3. Finally, in Section 3.4 we collect some local calculations.

3.1. Global theory

3.1.1. The regularized second fundamental form

Fix a compact n-manifold M. Let $\mathcal{M}$ denote the space of smooth Riemannian metrics on M, and let $\mathcal{D}$ denote the group of smooth diffeomorphisms of M. $\mathcal{D}$ acts on $\mathcal{M}$ by pullback: if $\psi \in \mathcal{D}$, $g \in \mathcal{M}$, then $\psi \cdot g = \psi^* g$. If we impose standard Sobolev norms on $\mathcal{M}$, $\mathcal{D}$, then $\mathcal{M}$ becomes a Banach manifold and $\mathcal{D}$ an ILH Lie group [16], and the action of $\mathcal{D}$ on $\mathcal{M}$ is as differentiable as desired. $\mathcal{D}$ is also a group before Sobolev norms are imposed, and once the norms are chosen, composition of diffeomorphisms produces a diffeomorphism also as differentiable as desired. We will assume that the choice of norms has been made.

Fix a metric g_0 on M. The orbit $\mathcal{O}_{g_0}$ through g_0 is diffeomorphic to $\mathcal{D}/S_{g_0}$, where S_{g_0} is the stabilizer of g_0. As in finite dimensions, it would be natural to assume that $S_{g_0} = \{\mathrm{id}\}$, so that the map $\psi \mapsto \psi^* g_0$ is an immersion of $\mathcal{D}$ in $\mathcal{M}$. To ensure that all orbits are of the same diffeomorphism type, we will assume instead that *the dimension of* S_g *is constant for all* g *near* g_0. To make the analogy with Section 1, we need Riemannian metrics on $\mathcal{M}, \mathcal{D}$. Now $\mathcal{M}$ comes with the standard L^2 inner product. Namely, $\mathcal{M}$ is an open cone in $\Gamma(S^2T^*M)$, the space of (sections of the) symmetric two-tensors on M, and the inner product of $h, k \in T_{g_0}\mathcal{M}$ is given by

$$\langle h, k\rangle_{g_0} = \int_M (g_0)^{ik}(g_0)^{jl}h_{ij}k_{kl}\,\mathrm{dvol}_{g_0},$$

where we follow the convention of writing a global integral in terms of a locally defined integrand. Here of course $g_0 = (g_0)_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ locally, and similarly for h, k, with $(g_0)^{ij}$ the inverse matrix to $(g_0)_{ij}$. (In contrast to the gauge theory case, where $\mathcal{M}$ is replaced by a space of connections, this metric is not flat.) $\mathcal{D}$ acts on $\mathcal{M}$ via isometries; the geometry of $\mathcal{M}$ and the quotient space $\mathcal{M}/\mathcal{D}$ is treated in [4,5].

To put a metric on $T_\psi\mathcal{D}$, it is sufficient to put an inner product on $T_{\mathrm{id}}\mathcal{D}$ and then left translate it to all of $\mathcal{D}$. However, $T_{\mathrm{id}}\mathcal{D} = \Gamma(TM)$ has no natural metric, although once g_0 is chosen it has the L^2 metric

$$\langle X, Y\rangle_{g_0} = \int_M (g_0)_{ij}X^iY^j\,\mathrm{dvol}_{g_0}$$

for $X = X^i\partial_i$, $Y = Y^i\partial_i$. We will also call this metric on $\mathcal{D}$ just g_0.

We now proceed as in finite dimensions. We consider a variation $F : \mathcal{D} \times (-\epsilon, \epsilon) \to \mathcal{M}$ with $F(\psi, 0) = \psi^*g_0$. We put the product metric $g_0 \oplus \mathrm{d}\alpha^2$ on $\mathcal{D} \times (-\epsilon, \epsilon)$ and set $L_\alpha = \mathrm{d}F(\mathrm{id}, \alpha) : \Gamma(TM) \to \Gamma(S^2T^*M)$. At the point $g_\alpha = F(\mathrm{id}, \alpha)$, define $\overline{L}^*_\alpha$, L^*_α by

$$\langle L_\alpha\omega, \eta\rangle_{g_\alpha} = \langle \omega, \overline{L}^*_\alpha\eta\rangle_{g_0} = \langle \omega, L^*_\alpha\eta\rangle_{g_\alpha}$$

for $\omega \in \Gamma(TM)$, $\eta \in \Gamma(S^2T^*M)$. Set $\overline{\Delta}_\alpha = \overline{L}^*_\alpha L_\alpha$, $\Delta_\alpha = L^*_\alpha L_\alpha$. Of course $\overline{\Delta}_0 = \Delta_0$. Note that since we must use the product metric as in finite dimensions, we cannot use the natural operator Δ_α, but are forced to use the non-natural $\overline{\Delta}_\alpha$. Our assumption on the stabilizer is equivalent to assuming that *dim ker* $\overline{\Delta}_\alpha$ *is independent of* α.

Following [1,14] we now define minimal orbits of metrics by means of (2.3). We let $\{\phi_i\}$ be a g_0-orthonormal basis of $L^2(TM)$ satisfying $\overline{\Delta}\phi_i = \Delta\phi_i = \lambda_i\phi_i$. As we will see in Corollary 3.2, $\overline{\Delta}_\alpha$ is elliptic, so such a basis exists for all α, and by standard techniques can be chosen to depend smoothly on α. We set the zeta function of $\overline{\Delta}_0$ to be

$$\overline{\zeta}(s) = \sum_{\substack{i \\ \lambda_i \neq 0}} \lambda_i^{-s}. \tag{3.1}$$

We similarly define $\overline{\zeta}_\alpha$ for $\overline{\Delta}_\alpha$. This converges for $\mathrm{Re}(s)$ sufficiently large, and has a meromorphic continuation to all of $\mathbb{C}$ with a regular value at zero; this follows in a well known way from the ellipticity of $\overline{\Delta}$ and the subsequent asymptotic expansion of its heat kernel. Note that in contrast to the finite dimensional case, the kernel of Δ need not be trivial. However, by ellipticity the dimension of the kernel of L and hence of Δ is finite, and is independent of $g \in \mathcal{O}_{g_0}$, since $L_{\rm id} = (\mathrm{d}\psi)^{-1} L_\psi \,\mathrm{d}\psi$, for $\psi : \mathcal{D} \to \mathcal{D}$ acting by left multiplication. It follows that $\mathcal{O}_{g_0}$ is always a submanifold of $\mathcal{M}$, and that $L_\psi^* L_\psi$ is isospectral to $L_{\rm id}^* L_{\rm id}$; in particular, $\zeta'(0)$ for the zeta functions associated to these operators is constant along orbits.

Definition. The *component of the trace of the second fundamental form in the direction* N for the orbit of a metric g_0 is defined to be

$$\mathrm{Tr}_N\,\mathrm{II} = \lim_{s\to 1}\left[-\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}_{g_0}(\mathrm{e}^{-t\overline{\Delta}} L^* \nabla_N L)\,\mathrm{d}t - \frac{\delta_N \overline{\zeta}(0)}{2(s-1)}\right]. \tag{3.2}$$

An orbit $\mathcal{O}_{g_0}$ is *minimal* if $\mathrm{Tr}_N\,\mathrm{II} = 0$ for all normal vectors N at g_0.

Remark. Here ∇ is the Levi-Civita connection for the L^2 metric on $\mathcal{M}$, defined as usual by

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle \end{aligned}$$

(cf. [4]). The term $\nabla_N L$ in (3.2) equals $(\mathrm{d}/\mathrm{d}\alpha)|_{\alpha=0} L_\alpha$ in a frame in which $\nabla_N = \delta_N$ (i.e. L is varying). Since we are taking the trace at g_0, we may replace $\overline{\Delta}$ by Δ in the integral. However, $\overline{\zeta}_\alpha(s) = \zeta_\alpha(s)$, the zeta function for Δ_α, only at $\alpha = 0$, so we cannot replace $\delta_N \overline{\zeta}(0)$ by $\delta_N \zeta(0)$. Note that (3.20) shows that $\overline{\zeta}_\alpha(0)$ is smooth in α under our assumption, so $\delta_N \overline{\zeta}(0)$ makes sense. As is shown below in (3.9), the last term in (3.2) subtracts off a possible pole from the first term, so we can also write

$$\mathrm{Tr}_N\,\mathrm{II} = \mathrm{FP}\left[-\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}_{g_0}(\mathrm{e}^{-t\Delta} L^* \nabla_N L)\,\mathrm{d}t|_{s=1}\right], \tag{3.3}$$

where FP denotes the finite part. Note that since $\mathcal{D}$ acts isometrically, $\mathrm{Tr}_N\,\mathrm{II} = 0$ for all normal vectors at g_0 iff the same is true at any $g \in \mathcal{O}_{g_0}$. This is clear for orbits of isometric actions in finite dimensions, and can be checked by directly examining the right-hand side of (3.2); an easier proof will be given below.

We now show that the right-hand side of (3.2) is always finite. We have

$$\begin{aligned} &-\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}_{g_0}(\mathrm{e}^{-t\Delta} L^* \nabla_N L)\,\mathrm{d}t \\ &= -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} \sum_i \langle L^* \nabla_N L\phi_i, \mathrm{e}^{-t\Delta}\phi_i\rangle_{g_0}\,\mathrm{d}t \\ &= -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} \sum_i \mathrm{e}^{-\lambda_i t}\langle L^* \nabla_N L\phi_i, \phi_i\rangle_{g_0}\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \mathrm{e}^{-\lambda_i t} \langle \nabla_N L_\alpha \phi_i, L_\alpha \phi_i \rangle_{g_\alpha} \, \mathrm{d}t |_{\alpha=0} \\
&= -\frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \mathrm{e}^{-\lambda_i t} N \langle L_\alpha \phi_i, L_\alpha \phi_i \rangle_{g_\alpha} \, \mathrm{d}t |_{\alpha=0} \qquad (3.4) \\
&= -\frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \mathrm{e}^{-\lambda_i t} \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \langle L_\alpha \phi_i, L_\alpha \phi_i \rangle_{g_\alpha} \, \mathrm{d}t \\
&= -\frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \mathrm{e}^{-\lambda_i t} \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \langle \overline{\Delta}_\alpha \phi_i, \phi_i \rangle_{g_0} \, \mathrm{d}t \\
&= -\frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \sum_i \langle (\delta \overline{\Delta}_\alpha) \mathrm{e}^{-\lambda_i t} \phi_i, \phi_i \rangle_{g_0} \, \mathrm{d}t \\
&= -\frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_{g_0} (\delta \overline{\Delta}_\alpha \cdot \mathrm{e}^{-t\Delta}) \, \mathrm{d}t \\
&= -\tfrac{1}{2} \overline{\zeta}_N(s), \qquad (3.5)
\end{aligned}
$$

where

$$
\overline{\zeta}_N(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_{g_0} (\delta \overline{\Delta} \mathrm{e}^{-t\Delta}) \, \mathrm{d}t = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_{g_0} (\delta \overline{\Delta} \mathrm{e}^{-t\overline{\Delta}}) \, \mathrm{d}t.
$$

Of course, this computation should be read as being valid for $\operatorname{Re}(s)$ sufficiently large, where the convergence of the integrals is easily established. For example, consider the term $\operatorname{Tr}_{g_0}(\delta\overline{\Delta}\mathrm{e}^{-t\Delta})$ in the last equation. The operator $\delta_N \overline{\Delta} \cdot \mathrm{e}^{-t\Delta}$ has kernel $(\delta_N \overline{\Delta})_x e(t, x, y)$, where e is the kernel of $\mathrm{e}^{-t\Delta}$, and so has a good asymptotic expansion as $t \to 0$ [6, Lemma 1.7.7]. Breaking the integral $\int_0^\infty$ into $\int_0^1 + \int_1^\infty$ and plugging in the asymptotic expansion into the first integral shows that this integral exists near zero for $\operatorname{Re}(s)$ sufficiently large. Also, since $\overline{\Delta} = \Delta$ at g_0, $\operatorname{Tr}(\delta_N \overline{\Delta} \cdot \mathrm{e}^{-t\Delta}) = \operatorname{Tr}(\overline{L}^* \delta_N L \cdot \mathrm{e}^{-t\overline{\Delta}} + \delta_N \overline{L}^* \cdot L \cdot \mathrm{e}^{-t\overline{\Delta}})$. Now the kernel of $L\mathrm{e}^{-t\overline{\Delta}}$ has exponential decay as $t \to \infty$, since $\ker \overline{\Delta} = \ker L$, and hence so does the kernel of $\delta_N \overline{L}^* \cdot L \cdot \mathrm{e}^{-t\overline{\Delta}}$. On $\ker L$, $\langle \overline{L}^* \delta_N L \cdot \mathrm{e}^{-t\overline{\Delta}} \phi, \phi \rangle = \langle \delta_N L \cdot \mathrm{e}^{-t\overline{\Delta}} \phi, L\phi \rangle = 0$, and so $\operatorname{Tr}_{g_0}(\delta_N \overline{L}^* \cdot L \cdot \mathrm{e}^{-t\overline{\Delta}})$ also has exponential decay as $t \to \infty$. Thus the integral exists at infinity. (By (2.4) and (3.5), definition (3.2) of the regularized trace agrees with the definition in [1, (3.5)].)

To proceed with the proof of the finiteness of (3.2), we note that by the Mellin transform

$$
\overline{\zeta}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_{g_0} (\mathrm{e}^{-t\overline{\Delta}} - P) \, \mathrm{d}t,
$$

where P is the orthogonal projection of $L^2(TM, g_0)$ onto the kernel of $\overline{\Delta}$; adding in this projection makes the integral finite near infinity.

Lemma 3.1 (cf. [14, Lemma 5.5]). *For all $s \in \mathbb{C}$,*

$$(s-1)\overline{\zeta}_N(s) = -\delta_N \overline{\zeta}(s-1). \tag{3.6}$$

At poles of $\overline{\zeta}_N(s)$, this equation is to be interpreted as saying that the poles of $\overline{\zeta}_N(s)$ coincide with the poles of $\overline{\zeta}(s)$ shifted by one.

Proof of Lemma 3.1. By the uniqueness of the meromorphic continuation of $\overline{\zeta}_N(s)$ and $\overline{\zeta}(s)$, it suffices to prove the equation for $\mathrm{Re}(s) \gg 0$. Under the assumption that dim ker $\overline{\Delta}$ is constant, we get

$$\begin{aligned}
(s-1)\overline{\zeta}_N(s) &= \frac{s-1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{Tr}((\delta_N \overline{\Delta}) \mathrm{e}^{-t\overline{\Delta}})\, \mathrm{d}t \\
&= -\frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2}\, \mathrm{Tr}(-t(\delta_N \overline{\Delta}) \mathrm{e}^{-t\overline{\Delta}})\, \mathrm{d}t \\
&= -\frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \delta_N\, \mathrm{Tr}(\mathrm{e}^{-t\overline{\Delta}})\, \mathrm{d}t \\
&= -\frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \delta_N\, \mathrm{Tr}(\mathrm{e}^{-t\overline{\Delta}} - P)\, \mathrm{d}t \\
&= -\delta_N \overline{\zeta}(s-1).
\end{aligned} \tag{3.7}$$

Here we have used (2.5) to write

$$\begin{aligned}
-t \cdot \mathrm{Tr}_{g_0}(\delta_N \overline{\Delta} \cdot \mathrm{e}^{-t\overline{\Delta}}) &= -t \sum_i \mathrm{e}^{-\lambda_i t} \langle \delta_N \overline{\Delta} \phi_i, \phi_i \rangle = -t \sum_i \mathrm{e}^{-\lambda_i t} \dot{\lambda}_i \\
&= \delta_N \left(\sum_i \mathrm{e}^{-\lambda_i t} \right) = \delta_N\, \mathrm{Tr}_{g_0}(\mathrm{e}^{-t\overline{\Delta}}). \qquad \square
\end{aligned} \tag{3.8}$$

Combining this lemma with (3.2) and 3.4), we get

$$\begin{aligned}
\mathrm{Tr}_N\, \mathrm{II} &= -\frac{1}{2} \lim_{s \to 1} \left[\frac{-\delta_N \overline{\zeta}(s-1)}{s-1} + \frac{\delta_N \overline{\zeta}(0)}{s-1} \right] \\
&= -\frac{1}{2} \lim_{s \to 0} \left[\frac{-\delta_N [\overline{\zeta}(0) + s\overline{\zeta}'(0) + \mathrm{O}(s^2)]}{s} + \frac{\delta_N \overline{\zeta}(0)}{s} \right] \\
&= \tfrac{1}{2} \delta_N \overline{\zeta}'(0).
\end{aligned} \tag{3.9}$$

In summary, by (3.3), (3.5) and (3.9), we have

$$\mathrm{Tr}_N\,\mathrm{II} = \tfrac{1}{2}\mathrm{FP}\,\overline{\zeta}_N(1) = \tfrac{1}{2}\delta_N\overline{\zeta}'(0). \tag{3.10}$$

It is standard to relate the right-hand side of (3.10) to the regularized volume element for $\mathcal{O}_{g_\alpha}$ at g_α. For ϕ_i fixed to be independent of α, the volume element to the orbit at g_α is formally

$$\sqrt{\det(\langle L_\alpha\phi_i, L_\alpha\phi_i\rangle_{g_\alpha})}\phi_1^* \wedge \phi_2^* \wedge \cdots$$
$$= \sqrt{\det(\langle\overline{\Delta}_\alpha\phi_i, \phi_i\rangle_{g_0})}\phi_1^* \wedge \phi_2^* \wedge \cdots \tag{3.11}$$
$$= \sqrt{\det(\langle\Delta_\alpha\phi_i, \phi_i\rangle_{g_\alpha})}\phi_1^* \wedge \phi_2^* \wedge \cdots. \tag{3.12}$$

Since the $\{\phi_i\}$ are g_0-orthonormal (and not g_α-orthornormal), it is heuristically plausible that the expression under the square root in (3.11) should give the determinant of $\overline{\Delta}_\alpha$, whereas the corresponding term in (3.12) should not be thought of as $\det\Delta_\alpha$. Using the Ray–Singer regularization of the determinant of a Laplacian-type operator, we define (the "Hodge star" of) the volume element to $\mathcal{O}_{g_\alpha}$ to be the non-natural $\exp(-\frac{1}{2}\overline{\zeta}'_\alpha(0))$, where $\overline{\zeta}_\alpha$ is the zeta function for $\overline{\Delta}$. In particular, (3.10) shows that an orbit is minimal iff it is minimal among all nearby orbits, provided we assume that all nearby orbits are of the same type (i.e. all nearby orbits have dim ker L_α = dim ker L_0, or equivalently are diffeomorphic to $\mathcal{O}_{g_0}$. The point here is that the zeta function behaves discontinuously in α if the dimension of the kernel jumps, so our analysis breaks down.) As in [14, Theorem 5.14], we interpret this as an infinite dimensional analogue of Hsiang's theorem in finite dimensions, which reduces the search for minimal orbits to checking variations only through orbits, and not through arbitrary submanifolds.

3.1.2. Comparing determinants

In order to produce examples of minimal orbits, we need to compare $\delta_N\overline{\zeta}'(0)$ with the more natural $\delta_N\zeta'(0)$ for two reasons: $\zeta'(0)$, although notoriously difficult to compute, can be handled in some special cases (cf. Theorem 3.2), and Bleecker's theorem about critical metrics applies to natural Lagrangians (cf. Theorem 3.5). Here $\zeta(s)$, $\zeta_\alpha(s)$ are defined in the usual way from the non-zero eigenvalues of Δ, Δ_α.

Looking back at (3.4), we get

$$\overline{\zeta}_N(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum \mathrm{e}^{-\lambda_i t}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0}\langle L\phi_i, L\phi_i\rangle_{g_\alpha}\,\mathrm{d}t$$
$$= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum \mathrm{e}^{-\lambda_i t}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0}\langle \Delta_\alpha, \phi_i, \phi_i\rangle_{g_\alpha}\,\mathrm{d}t.$$

Denote g_0 just by g. Since $(\mathrm{d}/\mathrm{d}\alpha)|_{\alpha=0}g_\alpha = N$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0}\langle\Delta_\alpha\phi_i, \phi_i\rangle_{g_\alpha}$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0} \int\limits_M (g_\alpha)_{ab}\phi_i^a(\Delta_\alpha\phi_i)^b \,\mathrm{dvol}(g_\alpha)\\
&= \int\limits_M N_{ab}\phi_i^a(\Delta_0\phi_i)^b \,\mathrm{dvol}(g_0) + \int\limits_M g_{ab}\phi_i^a((\delta\Delta_\alpha)\phi_i)^b \,\mathrm{dvol}(g_0)\\
&\quad + \int\limits_M g_{ab}\phi_i^a(\Delta_0\phi_i)^b(\mathrm{tr}_{g_0} N)\,\mathrm{dvol}(g_0)\\
&= \int\limits_M [g_{bs}g^{sc}N_{ca}\phi_i^a(\Delta_0\phi_i)^b + \langle\delta\Delta_\alpha\phi_i,\phi_i\rangle_{g_0}\\
&\quad + \langle\mathrm{tr}(N)\phi_i, \Delta\phi_i\rangle_{g_0}]\mathrm{dvol}(g_0).
\end{aligned}
\tag{3.13}
$$

Define the 0th order operator A on $\Gamma(TM)$ by $A : \phi^a\partial_a \mapsto g^{sc}N_{ca}\phi^a\partial_s$. Note that A lowers an index by N and raises an index by $g = g_0$.

Thus

$$
\begin{aligned}
\bar{\zeta}_N(s) &= \frac{1}{\Gamma(s)}\int\limits_0^\infty t^{s-1}\sum_i \mathrm{e}^{-\lambda_i t}\{\langle A\phi_i, \Delta\phi_i\rangle_{g_0} + \langle\delta\Delta_\alpha\phi_i,\phi_i\rangle_{g_0}\\
&\qquad\qquad + \langle\mathrm{tr}(N)\phi_i, \Delta\phi_i\rangle_{g_0}\}\,\mathrm{d}t\\
&= \frac{1}{\Gamma(s)}\int\limits_0^\infty t^{s-1}\sum_i\langle\phi_i, A^*\Delta\mathrm{e}^{-t\Delta}\phi_i\rangle_{g_0}\,\mathrm{d}t + \frac{1}{\Gamma(s)}\int\limits_0^\infty t^{s-1}\,\mathrm{Tr}_{g_0}(\delta\Delta_\alpha\mathrm{e}^{-t\Delta})\,\mathrm{d}t\\
&\quad + \frac{1}{\Gamma(s)}\int\limits_0^\infty t^{s-1}\sum_i\langle\phi_i, \mathrm{tr}(N)\Delta\mathrm{e}^{-t\Delta}\phi_i\rangle_{g_0}\,\mathrm{d}t.
\end{aligned}
\tag{3.14}
$$

Because $g^{sc}N_{ca} = N_a^s$ is a self-adjoint transformation of TM, A is self-adjoint. Explicitly, we have

$$
\langle A\phi, \psi\rangle = \int g_{bs}g^{sc}N_{ca}\phi^a\psi^b\,\mathrm{dvol}(g_0) = \int g_{aq}\phi^a(N_c^q\delta_b^c\psi^b)\,\mathrm{dvol}(g_0),
$$

so

$$
(A^*\psi)^q = N_c^q\delta_b^c\psi^b = N_b^q\psi^b = (A\psi)^q.
$$

Thus if we set

$$
\zeta_N(s) = \frac{1}{\Gamma(s)}\int\limits_0^\infty t^{s-1}\,\mathrm{Tr}(\delta\Delta_\alpha\mathrm{e}^{-t\Delta})\,\mathrm{d}t,
$$

(3.14) gives

$$
\bar{\zeta}_N(s) = \zeta_N(s) + \frac{1}{\Gamma(s)}\int\limits_0^\infty t^{s-1}\,\mathrm{Tr}(B\Delta\mathrm{e}^{-t\Delta})\,\mathrm{d}t,
$$

where B is the 0th order operator on TM given by

$$B\phi = A\phi + \mathrm{tr}_{g_0} N \cdot \phi.$$

It is easy to extract from (3.13) that B is characterized by

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} \langle \phi, \phi \rangle_{g_\alpha} = \langle B\phi, \phi \rangle_{g_0}. \tag{3.15}$$

Using $(\partial_t + \Delta)\mathrm{e}^{-t\Delta} = 0$ gives

$$\begin{aligned} \overline{\zeta}_N(s) &= \zeta_N(s) - \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{Tr}(B\partial_t \mathrm{e}^{-t\Delta})\, \mathrm{d}t \\ &= \zeta_N(s) + \frac{s-1}{\Gamma(s)} \int_0^\infty t^{s-2}\, \mathrm{Tr}(B(\mathrm{e}^{-t\Delta} - P))\, \mathrm{d}t \\ &= \zeta_N(s) + \frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2}\, \mathrm{Tr}(B\mathrm{e}^{-t\Delta} - BP)\, \mathrm{d}t. \end{aligned} \tag{3.16}$$

Recall that P denotes projection onto the kernel of $\overline{\Delta} = \Delta$; this term is added to make the integrals converge at infinity so that the integration by parts is valid. By the remarks after (3.2) and (3.5),

$$\begin{aligned} \mathrm{Tr}_N\, \mathrm{II} &= \tfrac{1}{2}\mathrm{FP}\overline{\zeta}_N(1) \\ &= \frac{1}{2}\mathrm{FP}\left(\zeta_N(1) + \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{Tr}(B\mathrm{e}^{-t\Delta} - BP)\, \mathrm{d}t|_{s=0} \right). \end{aligned} \tag{3.17}$$

We now analyse the last term in (3.17). Let $\mathrm{e}^{-t\Delta}$ have kernel $e(t, x, y) \in \Gamma(T_x M \otimes T_y M)$ with asymptotic expansion

$$e(t, x, x) \sim \sum_{k=0}^{\infty} t^{k-(n/2)} a_k(x, x) \quad \text{as } t \downarrow 0$$

($n = \dim M$). Then $B\mathrm{e}^{-t\Delta}$ has kernel $B_x e(t, x, y)$ where B_x means B acting in the x-variable. Thus

$$B_x e(t, x, y)|_{x=y} \sim \sum_k t^{k-n/2} B_x a_k(x, y)|_{x=y},$$

and so for $N \gg 0$,

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, \mathrm{Tr}(B\mathrm{e}^{-t\Delta} - BP)\, \mathrm{d}t|_{s=0}$$

$$= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\left(\sum_0^N t^{k-(n/2)} \int_M \operatorname{tr} B_x a_k \,\mathrm{dvol}(g_0) \right) + \mathrm{O}(t^N) - \operatorname{Tr}(BP) \right) \mathrm{d}t|_{s=0}$$

$$+ \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \operatorname{Tr}(B\mathrm{e}^{-t\Delta} - BP)\,\mathrm{d}t|_{s=0}$$

$$= \frac{1}{\Gamma(s)} \left(\sum_{k=0}^N \frac{\int_M \operatorname{tr} B_x a_k \,\mathrm{dvol}(g_0)}{k - n/2 + s} + \mathrm{O}(t^{N+k-n/2+s}) - \frac{\operatorname{Tr}(BP)}{s} \right) \Bigg|_{s=0} + 0$$

$$= \begin{cases} \int_M \operatorname{tr} B_x a_{n/2} \,\mathrm{dvol}(g_0) - \operatorname{Tr}(BP), & n \text{ even}, \\ -\operatorname{Tr}(BP), & n \text{ odd}. \end{cases} \tag{3.18}$$

In particular, the last term in (3.17) is always finite, so

$$\operatorname{Tr}_N \mathrm{II} = \frac{1}{2} \mathrm{FP}\, \bar{\zeta}_N(1) = \frac{1}{2} \left[\mathrm{FP}\, \zeta_N(1) + \int_M \operatorname{tr} B_x a_{n/2} \,\mathrm{dvol}(g_0) - \operatorname{Tr}(BP) \right],$$

with the understanding that the integral is zero in odd dimensions. As in (3.10), this shows that

$$\operatorname{Tr}_N \mathrm{II} = \frac{1}{2} \left[\delta_N \zeta'(0) + \int_M \operatorname{tr} B_x a_{n/2} \,\mathrm{dvol}(g_0) - \operatorname{Tr}(BP) \right]. \tag{3.19}$$

To sum up, in odd dimensions the non-local quantities $\delta_N \bar{\zeta}'(0)$, $\delta_N \zeta'(0)$ differ only by $\operatorname{Tr}(BP)$, and in even dimensions they differ by this term and the integral of a local expression.

Finally, we discuss the usual volume fixing conventions. As is clear from Lemma 3.2, under a scaling of the metric $g \mapsto \lambda^2 g$, we have $\Delta \mapsto \lambda^{-2}\Delta$. This implies that $\zeta'(0) \mapsto \zeta'(0) + 2\log\lambda \cdot \zeta(0)$. As in (3.18),

$$\zeta(0) = \begin{cases} \int_M \operatorname{tr} a_{n/2} \,\mathrm{dvol}(g_0) - \dim\ker\Delta, & n \text{ even}, \\ -\dim\ker\Delta, & n \text{ odd}. \end{cases} \tag{3.20}$$

Thus $\zeta'(0)$ is not scale invariant unless n is odd and we are in the "generic" case $\ker\Delta = 0$, which corresponds to M admitting no one-parameter family of isometries. So in general, we must restrict attention to infinitesimally volume preserving variations of the metric, or to those directions N with $\int_M \operatorname{tr}(N)\,\mathrm{dvol}(g_0) = 0$. These directions need not be normal to $\mathcal{O}_{g_0}$. However, writing $N = N^T + N^\nu$ in its tangential and normal components, we have $\delta_{N^T}\zeta'(0) = 0$ and so $\delta_N \zeta'(0) = \delta_{N^\nu}\zeta'(0)$. Thus we will restrict attention to normal variations which are projections of infinitesimally volume preserving variations N, and we still have that an orbit is minimal (among orbits with such variation vector field) iff $\delta_N\zeta'(0) = 0$. The easiest way to arrange this is to restrict attention to $\mathcal{M}_k$, the set of metrics on M of fixed volume k; $\mathcal{M}_k$ is a codimension one submanifold of $\mathcal{M}$.

There are topological conditions which force all orbits to be of generic type. Of course, the diffeomorphism type of an orbit $\mathcal{O}_{g_0}$ is determined by the stabilizer $S_{g_0} = \ker L$. This is the space of infinitesimal isometries of g_0, so $\dim\ker L$ equals the dimension of the space of isometries of g_0. If $\hat{A}(M) \neq 0$, then as noted in [9, p. 59], by a result of Atiyah–Hirzebruch M does not admit a circle action, much less a non-discrete Lie group of isometries. Moreover, if $p_1(M) = 0$, then there is an infinite sequence of characteristic numbers which are obstructions to M admitting a circle action [13].

3.2. *Minimal flat tori*

We will now determine two minimal orbits of flat 2-tori of fixed volume and show that one orbit is a stable minimum. This proceeds in two steps: first showing that we may use the natural $\zeta'(0)$ to compute $\mathrm{Tr}_N\,\mathrm{II}$, and then using the action of $SL(2,\mathbb{Z})$ on the space of tori to find critical metrics for $\zeta'(0)$. Finally, work of Montgomery [15] determines the flat metric for which $\zeta'(0)$ is minimal.

As we will see, the dimension of $\ker\overline{\Delta}$ is independent of the flat torus. This implies that we can use definition (3.2) to compute $\mathrm{Tr}_N\,\mathrm{II}$, since $\overline{\zeta}(0)$ is a smooth function of the tori. By (3.19), $\mathrm{Tr}_N\,\mathrm{II} = (1/2)[\delta_N\zeta'(0) - \mathrm{Tr}(BP)]$ provided

$$\int_M \mathrm{tr}\; B_x a_1 \,\mathrm{dvol}(g_0) = 0,$$

whenever g_0 is a flat metric on a torus. Of course the variation direction N contained in the definition of B must be infinitesimally volume preserving, i.e.

$$\int_M \mathrm{tr}_{g_0} N \,\mathrm{dvol}(g_0) = 0.$$

Note that in fact $\mathrm{tr}_{g_0} N = 0$; i.e. N is volume element preserving. For if the torus is given by the lattice spanned by $(1,0)$, (a,b), then the volume form for the coordinate chart $x \mapsto x + ay$, $y \mapsto by$ is $b\,\mathrm{d}x \wedge \mathrm{d}y$ and the volume of the torus is of course b. Thus the condition $\delta_N b = 0$ is equivalent to both volume preserving and volume form preserving.

Proposition 3.1. *On an n-manifold M,* $\mathrm{Tr}(B\mathrm{e}^{-t\Delta})$ *has an asymptotic expansion*

$$\mathrm{Tr}(B\mathrm{e}^{-t\Delta}) \sim \sum_{k=0}^{\infty} t^{k-(n/2)} \int_M b_k(x)\,\mathrm{dvol}.$$

On flat even dimensional manifolds, $b_{n/2}(x) \equiv 0$.

Of course, $b_k(x) = B_x a_k(x, y)|_{x=y}$, so the existence of the asymptotic expansion will follow from that for $e^{-t\Delta}$; this in turn is immediate from the ellipticity of Δ, which we will show in Section 3.4 along with the proof of Proposition 3.1.

Corollary 3.1. *For volume preserving variations of flat 2-tori,* $\mathrm{Tr}_N\,\mathrm{II} = (1/2)(\delta_N\zeta'(0) - \mathrm{Tr}(BP))$.

Proof. It is standard that

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}(Be^{-t\Delta})\,\mathrm{d}t|_{s=0} = \int_M b_m(x)\,\mathrm{dvol}$$

on a $2m$-manifold. By Proposition 3.1, b_1 vanishes on a flat torus. □

We next show that $\mathrm{Tr}(BP) = 0$. Complex structures on $S^1 \times S^1$ are parametrized by the upper half plane $\mathbb{H}$, where the point $\tau = (a, b)$ corresponds to the torus associated to the lattice generated by $(1, 0)$, (a, b). Two structures are isomorphic iff the lattices differ by an element of $SL(2, \mathbb{Z})$ or a homothety. Conversely, a flat torus comes from a lattice and so gives rise to a complex structure. However, a homothety of a lattice gives rise to a flat torus with a scaled metric, so these tori are not isometric. Thus tori of fixed volume are in one-to-one correspondence with $\mathbb{H}/SL(2, \mathbb{Z})$.

Fix $(a, b) \in \mathbb{H}$. For the associated torus $T^2_{(a,b)}$ with coordinate chart $[0, 1] \times [0, 1] \to T^2_{(a,b)}(x, y) \mapsto (x + ay, by)$, the metric takes the form

$$(g_{ij}) = \begin{pmatrix} 1 & a \\ a & a^2 + b^2 \end{pmatrix}$$

since $\partial_x = \boldsymbol{i}$, $\partial_y = a\boldsymbol{i}+b\boldsymbol{j}$, where $\{\boldsymbol{i},\boldsymbol{j}\}$ are the standard basis of $\mathbb{R}^2$. To show that $\mathrm{Tr}(BP) = 0$ for all tangent vectors N at (a, b), it suffices to consider the cases where N is a vertical vector or a horizontal vector. (We avoid naming these vectors to avoid confusion with ∂_x, ∂_y above.) Consider first a horizontal vector N. When we vary the metrics in the a direction, the variation two-tensor for the metrics is

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 2a \end{pmatrix}.$$

(The trace of N is non-zero because we are not working in an orthonormal frame at a point.) It is easily seen that the kernel of Δ is two-dimensional on $T^2_{(a,b)}$, since the group of isometries of $T^2_{(a,b)}$ consists of translations and possibly a discrete group of rotations. In particular, $\{\partial_x, \partial_y\}$ span $\ker\Delta$. The L^2 inner products of this basis are given by

$$\langle\partial_x, \partial_x\rangle = b, \qquad \langle\partial_x, \partial_y\rangle = ab, \qquad \langle\partial_y, \partial_y\rangle = (a^2 + b^2)b,$$

e.g.

$$\langle\partial_x, \partial_y\rangle = \int_{[0,1]\times[0,1]} g_{ij}(\partial_x)^i(\partial_y)^j\,\mathrm{dvol} = \int_{[0,1]\times[0,1]} g_{12}b\,\mathrm{d}x\,\mathrm{d}y = ab.$$

Thus an orthonormal basis of ker Δ is given by $\{b^{-1/2}\partial_x, b^{-3/2}(\partial_y - a\partial_x)\}$. As before $\mathrm{Tr}(BP)$ contains two terms, one of which as before involves tr N and so vanishes. The other term contributes

$$\begin{aligned}
\mathrm{Tr}(BP) &= \frac{1}{b}\langle B\partial_x, \partial_x\rangle + \frac{1}{b^3}\langle B(\partial_y - a\partial_x), \partial_y - a\partial_x\rangle \\
&= \frac{1}{b}\int\limits_{[0,1]\times[0,1]} g_{i1}g^{ic}N_{ca}\delta^{1a}\,\mathrm{dvol} \\
&\quad + \frac{1}{b^3}\int\limits_{[0,1]\times[0,1]} g_{i2}[g^{ic}N_{ca}\delta^{2a} - a(g^{ic}N_{ca}\delta^{1a})]\,\mathrm{dvol} \\
&\quad - \frac{a}{b^3}\int\limits_{[0,1]\times[0,1]} g_{i1}[g^{ic}N_{ca}\delta^{2a} - a(g^{ic}N_{ca}\delta^{1a})]\mathrm{dvol} \\
&= \frac{1}{b}\int\limits_{[0,1]\times[0,1]} N_{11} + \frac{1}{b^3}\int\limits_{[0,1]\times[0,1]} (N_{22} - aN_{12}) \\
&\quad - \frac{a}{b^3}\int\limits_{[0,1]\times[0,1]} (N_{12} - aN_{11}) \\
&= 0,
\end{aligned}$$

since $N_{11} = 0$.

For a vertical vector N, we not only alter b but also rescale the torus to fix the volume. Thus for $t \in [0, \epsilon]$ we consider the torus with chart given by $(x, y) \mapsto ((1-t)^{-1}(x + ay), (1-t)by)$; this is the torus of volume b associated to the point $(a, (1-t)^2 b)$. (We use $(1-t)^2$ rather than $1 - t$ to avoid square roots in the calculation.) Now $\partial_x = ((1-t)^{-1}, 0)$, $\partial_y = ((1-t)^{-1}a, (1-t)b)$, so

$$(g_{ij}) = \begin{pmatrix} (1-t)^{-1} & (1-t)^{-2}a \\ (1-t)^{-2}a & (1-t)^{-2}a^2 + (1-t)^2 b^2 \end{pmatrix}.$$

Thus

$$N = \begin{pmatrix} 2 & 2a \\ 2a & 2a^2 - 2b^2 \end{pmatrix}$$

and

$$\begin{aligned}
\mathrm{Tr}(BP) &= \frac{1}{b}\langle B\partial_x, \partial_x\rangle + \frac{1}{b^3}\langle B(\partial_y - a\partial_x), \partial_y - a\partial_x\rangle \\
&= \frac{1}{b}\int\limits_{[0,1]\times[0,1]} N_{11} + \frac{1}{b^3}\int\limits_{[0,1]\times[0,1]} (N_{22} - aN_{12}) \\
&\quad - \frac{a}{b^3}\int\limits_{[0,1]\times[0,1]} (N_{12} - aN_{11}) \\
&= 0.
\end{aligned}$$

These cancellations indicate that a second proof that $\mathrm{Tr}(BP) = 0$ can be obtained by mimicking the proof in Theorem 3.5, replacing G by $\mathbb{R}^2$, G/H by the torus, and using the G invariance of the flat metric. We leave this to the reader. In any case, we find that for volume preserving variations of flat tori,

$$\mathrm{Tr}_N \,\mathrm{II} = \tfrac{1}{2}\delta_N \zeta'(0). \tag{3.21}$$

This natural equation determines two flat tori whose orbits are minimal.

Theorem 3.1. *The orbits of the flat tori associated to the lattices* $(1, 0), (0, 1)$ *and* $(1, 0)$, $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ *are minimal within the space of all flat metrics of fixed volume.*

Proof. The points $(0, 1)$, $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ are the only points in the upper half plane with non-trivial stabilizer for the action of $SL(2, \mathbb{Z})$, and the stabilizer subgroups are isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3$ at these points, respectively [18, Ch. VII]. The differential of the action of a generator of the stabilizer groups therefore acts via rotation of π, $\frac{2}{3}\pi$ at the two fixed points. Since the action is by isometries, it must take the gradient vector of $\zeta'(0)$ to itself. Thus the gradient vector must vanish at these two points, i.e. $\delta_N \zeta'(0) = 0$. □

The proof above is a (trivial) example of Palais' symmetric criticality principle; Hsiang's theorem is a non-trivial example [17].

We can obtain more information about the orbit at $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ by computing $\zeta(s)$ in terms of the Epstein zeta function for the dual lattice of the torus. Recall that the dual lattice L^* to a lattice L in $\mathbb{R}^2$ is given by the set of $x^* \in \mathbb{R}^2$ such that $\langle x^*, x\rangle \in \mathbb{Z}$ for all $x \in L$. It is shown in [2, Ch. III.B] that the spectrum (with multiplicity) of the Laplacian $\Delta^0 = -\sum_{i=1}^{2}(\partial/\partial x_i)^2$ on functions on the torus is given by $\{4\pi^2|x^*|^2: x^* \in L^*\}$. If L^* is spanned by $\boldsymbol{a}^* = (1, 0), \boldsymbol{b}^* = (b_1, b_2)$, then for $x^* = m\boldsymbol{a}^* + n\boldsymbol{b}^*$, $|x^*|^2 = m^2 + 2b_1 mn = (b_1^2 + b_2^2)n^2 = f(m, n)$. Thus

$$\zeta_{\Delta^0}(s) = \left(\frac{1}{4\pi^2}\right)^s \sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)\neq(0,0)}} f(m,n)^{-s} = \left(\frac{1}{4\pi^2}\right)^s \zeta_{E,L^*}(s),$$

where the last term is by definition the Epstein zeta function of the lattice L^*.

Proposition 3.2. *Let* $\zeta_E(s) = \zeta_{E,L^*}(s)$ *denote the Epstein zeta function associated to the lattice* L^*. *Then the zeta function for* Δ *for the torus* T_L^2 *associated to the lattice* L *satisfies*

$$\zeta(s) = (1 + 2^{-s})(4\pi^2)^{-s}\zeta_E(s). \tag{3.22}$$

Proof. The eigenfunctions for Δ^0 are given by

$$f_{x^*}(y) = \mathrm{e}^{2\pi\langle x^*, y\rangle}$$

for $x^* \in L^*$ [2, Ch. III.B]. Thus any function $f \in L^2(T_L^2)$ can be expressed as

$$f(y) = \sum_{x^*\in L^*} a_{x^*} f_{x^*}(y).$$

A smooth 1-form $u = \sum_{i=1}^{2} u_i(x)\,\mathrm{d}x^i$ has the decomposition

$$u_1(y) = \sum_{x^* \in L^*} a_{x^*} f_{x^*}(y), \qquad u_2(y) = \sum_{x^* \in L^*} b_{x^*} f_{x^*}(y). \tag{3.23}$$

By the local expression for Δ given in Corollary 3.2, the eigenvalue equation for Δ becomes

$$(\Delta u)_j = -\left[\sum_{i=1}^{2} \frac{\partial^2 u_j}{\partial x_i^2} + \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i}\right)\right] = \lambda u_j. \tag{3.24}$$

Substituting (3.23) into (3.24) yields

$$\sum_{x^* \in L^*} 4\pi^2(|x^*|^2 \alpha_1^2 a_{x^*} + \alpha\beta b_{x^*}) f_{x^*} = \lambda \sum a_{x^*} f_{x^*},$$

$$\sum_{x^* \in L^*} 4\pi^2(|x^*|^2 b_{x^*} + \alpha\beta a_{x^*} + \beta^2 b_{x^*}) f_{x^*} = \lambda \sum b_{x^*} f_{x^*},$$

where $x^* = (\alpha, \beta)$. Setting $\lambda/4\pi^2 = \mu$, we find that the eigenvalue λ satisfies

$$\begin{vmatrix} |x^*|^2 - \alpha^2 - \mu & \alpha\beta \\ \alpha\beta & |x^*|^2 - \beta^2 - \mu \end{vmatrix} = 0.$$

Thus we have

$$\mu = |x^*|^2,\ 2|x^*|^2.$$

Note that the zero eigenforms of Δ form a two-dimensional space, agreeing with our earlier computation.

In conclusion, the zeta function of Δ is given by

$$\begin{aligned} \zeta(s) &= \sum_{x^* \in L^* - \{(0,0)\}} \left(\frac{1}{|x^*|^2 4\pi^2}\right)^s + \left(\frac{1}{8\pi^2 |x^*|^2}\right)^s \\ &= \left(1 + \frac{1}{2^s}\right)(4\pi^2)^{-s} \zeta_E(s). \qquad \square \end{aligned}$$

The value of $\zeta_E'(0)$ is given in terms of the Dedekind eta function [11, Ch. 20]. However, it seems difficult to determine lattices for which $\zeta_E'(0)$ is critical this way.

Since the volume element is formally given by the (square root of the) determinant of $\overline{\Delta}$, by (3.10) and (3.21), it is natural to measure the stability of a minimal orbit by the second variation $\delta^2_{N,M} \det \overline{\Delta}$. For flat tori, by (3.10) and (3.19), $\overline{\zeta}'(0)$ and $\zeta'(0)$ differ by a constant, so we may measure stability by $\delta^2_{N,M} \det \Delta$. We will say that a minimal orbit is stable if $\delta^2_{N,M} \zeta'(0) \geq 0$ for all N, M.

Theorem 3.2. *The orbit of the flat torus associated to* $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ *is stable within the space of flat tori of fixed volume.*

Proof. Set $\xi(s) = \zeta_E(s)\Gamma(s)(2\pi)^{-s}$. By [15, p. 75], $\xi(s)$ has a minimum at the torus associated to $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ for all $s \in (0, \frac{1}{2})$. (This uses the fact that $L^* = L$ for this lattice.) Thus $0 \le \delta^2_{N,M}\xi(s)$ for all $s \in (0, \frac{1}{2})$. Substituting (3.22) for $\zeta_E(s)$ gives

$$\begin{aligned} 0 &\le \frac{(4\pi)^s}{2^s+1}\delta^2_{N,M}(\zeta(s)\Gamma(s)) \\ &= \frac{(4\pi)^s}{2^s+1}(\delta^2_{N,M}\zeta(0) + s\delta^2_{N,M}\zeta'(0) + \mathrm{O}(s^2))\left(\frac{1}{s} + 1 + \mathrm{O}(s^2)\right) \\ &= \frac{(4\pi)^s}{2^s+1}(\delta^2_{N,M}\zeta'(0) + \mathrm{O}(s)). \end{aligned}$$

Here we have used $\zeta(0) = -\dim\ker\Delta = -2$ for all flat tori. Letting s go to zero gives $0 \le \delta^2_{N,M}\zeta'(0)$. □

The same argument for first variations gives another proof that this torus gives a minimal orbit.

3.3. Orbits of varying type

While the case of generic orbits (and more generally families of orbits of fixed type) treated in Section 3.1 is easiest to handle, minimal orbits often occur outside these cases. In particular, in finite dimensions, it is an easy corollary of Hsiang's theorem that orbits of isolated diffeomorphism type are minimal submanifolds. The proof uses the exponential map, which may not be available in our context. We now discuss how to handle orbits of varying type in infinite dimensions. The main results are that orbits of isolated type are minimal (Theorem 3.3) and that isotropy irreducible homogeneous spaces with invariant metrics are minimal (Theorem 3.5).

Let $\overline{\lambda} = \overline{\lambda}_0$ be the first non-zero eigenvalue of $\overline{\Delta}_0$. There is a neighbourhood of 0 in $(T_{g_0}\mathcal{O}_{g_0})^\perp$ such that $\frac{1}{2}\overline{\lambda}$ is not in the spectrum of $\overline{\Delta}_{g_0+N}$ for all $N \in U$. For $g_\alpha = g_0 + \alpha N$ ($N \in U, \alpha \in [0, 1]$), let $\overline{P} = \overline{P}_\alpha$ be g_α-orthogonal projection into the sum of the eigenspaces of $\overline{\Delta}_\alpha$ with eigenvalues less than $\frac{1}{2}\overline{\lambda}$. Following [14, (5.17)], for $T > 1$ set

$$\begin{aligned} \overline{\zeta}_T(s) &= \overline{\zeta}_{T,\alpha}(s) \\ &= \frac{1}{\Gamma(s)}\int_0^T t^{s-1}\,\mathrm{Tr}_{g_0}(\mathrm{e}^{-t\overline{\Delta}_\alpha})\,\mathrm{d}t + \frac{1}{\Gamma(s)}\int_T^\infty t^{s-1}\,\mathrm{Tr}_{g_0}(\mathrm{e}^{-t(\overline{\Delta}_\alpha - \overline{P}_\alpha)})\,\mathrm{d}t. \end{aligned}$$

Both integrals are now smooth functions of α. Note that $\overline{\zeta}_T(s) = \overline{\zeta}(s)$ if dim ker $\overline{\Delta}$ is constant near g_0.

We have

$$\delta_N\overline{\zeta}_T(s-1) = \delta_N\overline{\zeta}_T(s-1) + \frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N\,\mathrm{Tr}(\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N \operatorname{Tr}(\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t \\
&=\frac{1}{\Gamma(s-1)}\int_0^\infty t^{s-2}\delta_N \operatorname{Tr}(\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t \\
&\quad+\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N \operatorname{Tr}(\mathrm{e}^{-t(\overline{\Delta}-\overline{P})})\,\mathrm{d}t \\
&\quad-\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N \operatorname{Tr}(\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t \\
&=-(s-1)\overline{\zeta}_N(s)+\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N \operatorname{Tr}(\mathrm{e}^{-t(\overline{\Delta}-\overline{P})})\,\mathrm{d}t \\
&\quad-\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N \operatorname{Tr}(\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t.
\end{aligned}
$$

Here we have used the part of (3.7) which does not assume that the kernel has constant dimension, and we recall that $\delta_N \operatorname{Tr}(\mathrm{e}^{-t\overline{\Delta}})$ has exponential decay at infinity by the remarks after (3.5). Thus

$$
\delta_N\overline{\zeta}_T(s-1)=-(s-1)\overline{\zeta}_N(s)-\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N \operatorname{Tr}_{g_0}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t. \tag{3.25}
$$

Moreover, the last term in $\overline{\zeta}_T(s)$ is zero at $s=0$, while plugging in the asymptotics for $\operatorname{Tr}(\mathrm{e}^{-t\overline{\Delta}})$ into the first integral yields

$$
\overline{\zeta}_T(0)=\overline{\zeta}(0)+\dim\ker\overline{\Delta}=\begin{cases}\displaystyle\int_M \operatorname{tr} a_{n/2}, & n \text{ even},\\ 0, & n \text{ odd}.\end{cases}
$$

We now extend the definition of $\operatorname{Tr}_N \mathrm{II}$ for orbits of arbitrary type.

Definition.

$$
\begin{aligned}
\operatorname{Tr}_N \mathrm{II}&=\lim_{s\to 1}\left[-\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}_{g_0}(\mathrm{e}^{-t\overline{\Delta}}L^*\nabla_N L)\,\mathrm{d}t-\frac{\delta_N\overline{\zeta}_T(0)}{2(s-1)}\right]\\
&=\lim_{s\to 1}\left[-\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}_{g_0}(\mathrm{e}^{-t\overline{\Delta}}L^*\nabla_N L)\,\mathrm{d}t+\frac{\delta_N(\overline{\zeta}(0)+\dim\ker\overline{\Delta})}{2(s-1)}\right].
\end{aligned}
$$

As before, we may replace $\overline{\Delta}$ with Δ. Since (3.4) is unchanged, by (3.25) we get

$$\begin{aligned}\mathrm{Tr}_N\,\mathrm{II} &= -\frac{1}{2}\lim_{s\to 1}\left[\frac{-\delta_N\overline{\zeta}_T(s-1)}{s-1}\right.\\ &\qquad \left. + \frac{1}{(s-1)\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N\,\mathrm{Tr}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t + \frac{\delta_N\overline{\zeta}_T(0)}{s-1}\right]\\ &= \frac{1}{2}\delta_N\overline{\zeta}'_T(0) - \frac{1}{2}\int_T^\infty t^{-1}\delta_N\,\mathrm{Tr}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t. \qquad (3.26)\end{aligned}$$

(It is shown in [14, p. 200] that $\delta_N\,\mathrm{Tr}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})$ has exponential decay at infinity.)

Theorem 3.3. *An orbit $\mathcal{O}_{g_0}$ of isolated diffeomorphism type is minimal.*

Proof. The isometric action of the stabilizer S_{g_0} on $\mathcal{M}$ induces an action on $T_{g_0}\mathcal{O}_{g_0}$, still given by $\phi\cdot v = \phi^* v$, which is easily seen to be unitary. Thus S_{g_0} acts unitarily on $X = (T_{g_0}\mathcal{O}_{g_0})^\perp$.

Since M is compact, S_{g_0} is a compact Lie group, so X splits into a sum of finite dimensional irreducible representations X_i of S_{g_0}. On each piece we can define $\mathrm{Tr}\,\mathrm{II}|_{X_i} = \sum_j(\mathrm{Tr}_{N_j}\,\mathrm{II})N_j$, where $\{N_j\}$ is an orthonormal basis of X_i. $\mathrm{Tr}\,\mathrm{II}|_{X_i}$ is of course independent of the choice of this basis, so for all $\phi\in S_{g_0}$, $\mathrm{Tr}\,\mathrm{II}|_{X_i} = \sum_j(\mathrm{Tr}_{\mathrm{d}\phi(N_j)}\,\mathrm{II})\,\mathrm{d}\phi(N_j)$. But by (3.26) $\mathrm{Tr}_N\,\mathrm{II} = \mathrm{Tr}_{\mathrm{d}\phi(N)}\,\mathrm{II}$, since $\overline{\Delta}$ stays isospectral under the action of ϕ (cf. [14, (5.23)]). Thus $\mathrm{Tr}\,\mathrm{II}|_{X_i}$ is fixed by S_{g_0}: i.e. $\mathrm{Tr}\,\mathrm{II}|_{X_i} = \mathrm{d}\phi(\mathrm{Tr}\,\mathrm{II}|_{X_i})$.

If $\mathcal{O}_{g_0}$ is not minimal, then $\mathrm{Tr}_{N_0}\,\mathrm{II}\neq 0$ for some $N_0\in X$, and hence on some neighbourhood of N_0 in X. The vector space spanned by the $\{N_j\}$, $i = 1, 2, \ldots$, is dense in X, and so $\mathrm{Tr}\,\mathrm{II}|_{X_i}\neq 0$ for some i. (If we knew that $N\mapsto \mathrm{Tr}_N\,\mathrm{II}$ were continuous in N, then from $\mathrm{Tr}_{N_0}\,\mathrm{II}\neq 0$ we could directly conclude $\mathrm{Tr}\,\mathrm{II}|_{X_i}\neq 0$ for some i.)

Set $\mathrm{Tr}\,\mathrm{II}|_{X_i} = A$. We now claim that the orbits $\mathcal{O}_{g_0+\epsilon A}$ are diffeomorphic to $\mathcal{O}_{g_0}$ for all $\epsilon > 0$; this contradicts the isolation of $\mathcal{O}_{g_0}$. Note that for all $\phi\in\mathcal{D}$

$$\mathrm{d}\phi(A) = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0}\phi^*(g_0+\alpha A) = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0}\phi^* g_0 + \alpha\phi^* A = \phi^* A,$$

so $\phi^*(g_0+\epsilon A) = \phi^* g_0 + \epsilon\,\mathrm{d}\phi(A)$. Thus $\mathcal{O}_{g_0+\epsilon A} = \mathcal{O}_{\phi^* g_0+\epsilon\phi(A)}$. Also, since $\mathrm{d}\phi(A) = A$, we have $\phi^*(g_0+\epsilon A) = g_0+\epsilon A$ iff $\phi\in S_{g_0}$. Thus $\mathcal{O}_{g_0+\epsilon A}\approx \mathcal{D}/S_{g_0}\approx\mathcal{O}_{g_0}$. □

Theorem 3.4. *The orbits of the following metrics are minimal within the space of metrics of fixed volume:*

(i) *the standard metric g_0 on S^n;*
(ii) *the standard metric g_1 on $\mathbb{RP}^n$.*

Consider the product metric $ds^2 \oplus g_0$ *on* $S^1 \times S^n$, *where* $\int_{S^1} \mathrm{d}s^2 = 2\pi$. *Consider the space* $\mathcal{M}'$ *of all metrics on* $S^1 \times S^n$ *except diffeomorphism orbits of metrics* $\epsilon\, \mathrm{d}s^2 \oplus \epsilon^{1/n} g_0$, *for* $\epsilon > 0$. *Then the orbit of* $\mathrm{d}s^2 \oplus g_0$ *is minimal within* $\mathcal{M}'$. *The same statement holds for* $\mathrm{d}s^2 \oplus g_1$ *on* $S^1 \times \mathbb{RP}^n$.

Proof. S^n and $\mathbb{RP}^n$ with their standard metrics are the only compact n-manifolds of fixed volume with isometry group of dimension $n(n+1)/2$ [9, Theorem 3.1]. Thus these orbits are isolated. $S^1 \times S^n$, $S^1 \times \mathbb{RP}^n$, with metrics $\alpha\, \mathrm{d}s^2 \oplus \beta g_0$, $\alpha\, \mathrm{d}s^2 \oplus \beta g_1$, are the only compact $(n+1)$-manifolds with isometry groups of dimension $\frac{1}{2}n(n+1)+1$ [9, Theorem 3.3]. If we set $\alpha = \beta = 1$ and exclude other metrics of the same volume, then these orbits are minimal. □

We now produce a much larger list of minimal orbits (including the standard metrics on symmetric spaces) by replacing the non-natural $\overline{\zeta}_T(s)$ with its natural analogue.

Let λ_0 be the first non-zero eigenvalue of Δ_0, and let $P = P_\alpha$ denote g_α-orthogonal projection into the eigenspaces of Δ_α lying below $\frac{1}{2}\lambda_0$. Set

$$\begin{aligned}\zeta_T(s) &= \zeta_{T,\alpha}(s)\\ &= \frac{1}{\Gamma(s)} \int_0^T t^{s-1} \operatorname{Tr}_{g_\alpha}(\mathrm{e}^{-t\Delta_\alpha})\, \mathrm{d}t + \frac{1}{\Gamma(s)} \int_T^\infty t^{s-1} \operatorname{Tr}_{g_\alpha}(\mathrm{e}^{-t(\Delta_\alpha - P_\alpha)})\, \mathrm{d}t.\end{aligned}$$

For α close to zero, both terms on the right-hand side are smooth in α. By (3.16) and (3.25), we get

$$\begin{aligned}\delta_N \zeta_T(s-1) &= -(s-1)\zeta_N(s) - \frac{1}{\Gamma(s-1)} \int_T^\infty t^{s-2} \delta_N \operatorname{Tr}_{g_\alpha}(P_\alpha \mathrm{e}^{-t\Delta_\alpha})\, \mathrm{d}t\\ &= -(s-1)\left[\overline{\zeta}_N(s) - \frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \operatorname{Tr}_{g_0}(B\mathrm{e}^{-t\Delta} - BP)\, \mathrm{d}t\right]\\ &\quad - \frac{1}{\Gamma(s-1)} \int_T^\infty t^{s-2} \delta_N \operatorname{Tr}_{g_\alpha}(P_\alpha \mathrm{e}^{-t\Delta_\alpha})\, \mathrm{d}t\\ &= \delta_N \overline{\zeta}_T(s-1) + \frac{s-1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \operatorname{Tr}_{g_0}(B\mathrm{e}^{-t\Delta} - BP)\, \mathrm{d}t\\ &\quad + \frac{1}{\Gamma(s-1)} \int_T^\infty t^{s-2} \delta_N [\operatorname{Tr}_{g_0}(\overline{P}_\alpha \mathrm{e}^{-t\overline{\Delta}_\alpha}) - \operatorname{Tr}_{g_\alpha}(P_\alpha \mathrm{e}^{-t\Delta_\alpha})]\, \mathrm{d}t.\end{aligned}$$

Thus by (3.26) (and omitting the α's),

$$\begin{aligned}\mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{2}\lim_{s\to 1}\Bigg[&\frac{-\delta_N\zeta_T(s-1)}{s-1}\\&+\frac{1}{\Gamma(s-1)}\int_0^\infty t^{s-2}\,\mathrm{Tr}_{g_0}(B\mathrm{e}^{-t\Delta}-BP)\,\mathrm{d}t+\frac{\delta_N\overline{\zeta}_T(0)}{s-1}\Bigg]\\&-\frac{1}{2}\int_T^\infty t^{-1}\delta_N[\mathrm{Tr}_{g_0}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})-\mathrm{Tr}_{g_\alpha}(P\mathrm{e}^{-t\Delta})]\,\mathrm{d}t\\&-\frac{1}{2}\int_T^\infty t^{-1}\delta_N\,\mathrm{Tr}_{g_0}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t.\end{aligned}\tag{3.27}$$

We now make the third term in the above limit more natural. We have

$$\begin{aligned}\frac{\delta_N\overline{\zeta}_T(0)}{s-1}&=\frac{1}{s-1}\Bigg[-(s-1)\overline{\zeta}_N(s)|_{s=1}\\&\qquad-\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N\,\mathrm{Tr}_{g_0}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t|_{s=1}\Bigg]\\&=\frac{1}{s-1}\Bigg[-(s-1)\zeta_N(s)|_{s=1}\\&\qquad-\frac{(s-1)}{\Gamma(s-1)}\int_0^\infty t^{s-2}\,\mathrm{Tr}_{g_0}(B\mathrm{e}^{-t\Delta}-BP)\,\mathrm{d}t|_{s=1}\\&\qquad-\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_n\,\mathrm{Tr}_{g_0}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t|_{s=1}\Bigg].\end{aligned}$$

Now we know $(1/\Gamma(s)\int_0^\infty t^{s-1}\,\mathrm{Tr}(B\mathrm{e}^{-t\Delta}-BP)\,\mathrm{d}t|_{s=0}$ is finite, so

$$\frac{s-1}{\Gamma(s-1)}\int_0^\infty t^{s-2}\,\mathrm{Tr}(B\mathrm{e}^{-t\Delta}-BP)\,\mathrm{d}t|_{s=1}=0.$$

Thus by (3.25) for $\zeta_T(s)$,

$$\frac{\delta_N\overline{\zeta}_T(0)}{s-1}=\frac{1}{s-1}\left[\delta_N\zeta_T(s-1)|_{s=1}+\frac{1}{\Gamma(s-1)}\int_T^\infty t^{s-2}\delta_N\,\mathrm{Tr}_{g_\alpha}(P\mathrm{e}^{-t\Delta})\,\mathrm{d}t|_{s=1}\right]$$

$$
\begin{aligned}
&- \int_T^\infty t^{-1} \delta_N \operatorname{Tr}_{g_0}(\overline{P} e^{-t\overline{\Delta}})\, dt \\
&= \frac{\delta_N \zeta_T(0)}{s-1} + \int_T^\infty t^{-1} \delta_N \operatorname{Tr}_{g_\alpha}(P e^{-t\Delta})\, dt - \int_T^\infty t^{-1} \delta_N \operatorname{Tr}_{g_0}(\overline{P} e^{-t\overline{\Delta}})\, dt.
\end{aligned}
$$

Substituting this into (3.27) yields

$$
\begin{aligned}
\operatorname{Tr}_N \mathrm{II} = &-\frac{1}{2} \lim_{s\to 1} \left[\frac{-\delta_N \zeta_T(s-1)}{s-1} + \frac{\delta_N \zeta_T(0)}{s-1} \right] \\
&- \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_{g_0}(B e^{-t\Delta} - BP)\, dt|_{s=0} \\
&- \frac{1}{2} \int_T^\infty t^{-1} \delta_N \operatorname{Tr}_{g_0}(\overline{P} e^{-t\overline{\Delta}})\, dt.
\end{aligned}
$$

Plugging in the Taylor series for $\zeta_T(s-1)$ near $s = 1$ as before gives

$$
\begin{aligned}
\operatorname{Tr}_N \mathrm{II} = &\frac{1}{2} \delta_N \zeta_T'(0) - \frac{1}{2\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_{g_0}(B e^{-t\Delta} - BP)\, dt|_{s=0} \\
&- \frac{1}{2} \int_T^\infty t^{-1} \delta_N \operatorname{Tr}_{g_0}(\overline{P} e^{-t\overline{\Delta}})\, dt. \qquad (3.28)
\end{aligned}
$$

Recall that a homogeneous space G/H is called isotropy irreducible if the linearized isotropy representation of the identity component of H on $T_{[\mathrm{Id}]}(G/H)$ is irreducible. A complete list of simply connected examples other than symmetric spaces was given by Manturov, Wolf and Kraemer, see e.g. [10,19].

Theorem 3.5. *Let $M = G/H$ be an odd dimensional simply connected isotropy irreducible homogeneous space with its G-invariant metric g_0 of volume 1. Then $\mathcal{O}_{g_0}$ is minimal within the space of all volume one metrics.*

Proof. In odd dimensions, we have by (3.18) and (3.28)

$$
\operatorname{Tr}_N \mathrm{II} = \frac{1}{2} \delta_N \zeta_T'(0) + \frac{1}{2} \operatorname{Tr}(BP) - \frac{1}{2} \int_T^\infty t^{-1} \delta_N \operatorname{Tr}(\overline{P} e^{-t\overline{\Delta}})\, dt.
$$

Now $\zeta_{T,\alpha}(s)$ is a natural Lagrangian in the sense that it depends naturally on the metric g_α except for the term P_α, which depends on the non-natural λ_0. However, thinking of λ_0 as just a universal constant shows that $\zeta_{T,\alpha}(s)$ is a smooth natural Lagrangian for metrics g_α near g_0. (It will fail to be smooth for metrics on manifolds with $\frac{1}{2}\lambda_0$ in the spectrum of Δ.) This is enough to apply Bleecker's theorem [3] that invariant metrics on isotropy irreducible homogeneous spaces are critical for natural Lagrangians. (In brief, the gradient vector for this Lagrangian at g_0 will be a G-invariant symmetric two-tensor on G/H and so by hypothesis will be a multiple of the metric. Since we consider metrics of fixed volume, this multiple must be zero.) We conclude that $\delta_N \zeta_T'(0) = 0$ for all $N \in (T_{g_0}\mathcal{O}_{g_0})^{\perp}$. Thus for these N,

$$\mathrm{Tr}_N \,\mathrm{II} = \frac{1}{2}\mathrm{Tr}(BP) - \frac{1}{2}\lim_{T\to\infty}\int_T^{\infty} t^{-1}\delta_N \,\mathrm{Tr}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})\,\mathrm{d}t = \frac{1}{2}\mathrm{Tr}(BP).$$

We now show that $\mathrm{Tr}(BP) = 0$. By [19, Theorem 17.1], the identity component of the isometries of G/H is given by multiplication of cosets by elements of G. (For $S^7 = Spin(7)/G_2 = SO(8)/SO(7)$, we choose $G = SO(8)$.) If $\{p_i\}$ is an orthonormal basis of $\mathfrak{g}$, the Lie algebra of G, then ker Δ is spanned by $\{P_i\}$, where $(P_i)_{gH} = (\mathrm{d}/\mathrm{d}t)|_{t=0}(\exp_G tp_i)gH$. This basis is L^2-orthonormal, as

$$\begin{aligned}\langle P_i, P_j\rangle &= \int_{G/H} \langle(\mathrm{d}/\mathrm{d}t)|_{t=0}(\exp_G tp_i)gH, (\mathrm{d}/\mathrm{d}t)|_{t=0}(\exp_G tp_j)gH\rangle_{gH}\,\mathrm{dvol}\\ &= \int_{G/H} \langle(\mathrm{d}/\mathrm{d}t)|_{t=0}(\exp_G tp_i), (\mathrm{d}/\mathrm{d}t)|_{t=0}(\exp_G tp_j)\rangle_{H}\,\mathrm{dvol}\\ &= \int_{G/H} \langle p_i, p_j\rangle\,\mathrm{dvol} = \delta_{ij}\end{aligned}$$

by the G-invariance of the metric.

$\mathrm{Tr}(BP)$ has two terms, one of which is the trace of the pointwise multiplication by $\mathrm{tr}(N)$. This term contributes

$$\int_{G/H}\sum_i \langle \mathrm{tr}(N)P_i, P_i\rangle\,\mathrm{dvol} = (\dim \mathfrak{g})\int_{G/H}\mathrm{tr}(N)\,\mathrm{dvol} = 0,$$

since N is infinitesimally volume preserving. The second term A contributes

$$\int_{G/H}\sum_i g_{lm}g^{lc}N_{ca}\phi_i^a\phi_i^m\,\mathrm{dvol}$$

if $P_i = \phi_i^a\partial_a$ locally. At a point we can of course take $\partial_a = P_a$, in which case $g_{ij} = \delta_{ij}$, $P_i^a = \delta_i^a$ and the integrand becomes $\sum_i N_{ii} = \mathrm{tr}(N)$. Thus the second term also contributes zero. □

Remark. From the proof we see that $0 = \mathrm{Tr}_N\,\mathrm{II} = -\frac{1}{2}\int_T^{\infty} t^{-1}\delta_N\,\mathrm{Tr}(\overline{P}\mathrm{e}^{-t\overline{\Delta}})$, which implies that $\delta_N\,\mathrm{Tr}(\overline{P}\mathrm{e}^{-t\overline{\Delta}}) = 0$. Thus g_0 is critical even for this non-natural Lagrangian.

3.4. Local computations

In this section we will produce the asymptotic expansion for $\mathrm{e}^{-t\Delta}, \mathrm{e}^{-t\overline{\Delta}}$. Of course the existence of the asymptotic expansion is immediate once we check that $\Delta, \overline{\Delta}$ are elliptic (Corollary 3.2).

So fix a metric g_0 with associated Levi-Civita connection ∇ and Ricci curvature tensor $R_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$; we raise and lower indices using g_0. Pick $u = u^i\partial_i \in \Gamma(TM), \omega = \omega_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b \in \Gamma(S^2T^*M)$. Recall that we compute $\overline{L}, \overline{\Delta}$ with respect to another metric g. Set $\mu_0 = \sqrt{\det g_0}, \mu = \sqrt{\det g}$.

Lemma 3.2. *In local coordinates we have*

$$\begin{aligned}
(Lu)_{ab} &= (\nabla_a u)_b + (\nabla_b u)_a,\\
(L^*\omega)^a &= -2\nabla^b\omega^a_b,\\
(\overline{L}^*\omega)^a &= -2(g_0)^{au}\frac{\mu}{\mu_0}\nabla^b\omega_{bu},\\
(\Delta u)^a &= -2((\nabla^b\nabla_b u)^a + (\nabla^a\nabla_b u)^b + R^a_b u^b),\\
(\overline{\Delta} u)^a &= -2(g_0)^{as}\frac{\mu}{\mu_0}(\nabla^b\nabla_b u)_s - 2(g_0)^{as}\frac{\mu}{\mu_0}(\nabla_s\nabla^b u)_b + 2(g_0)^{as}\frac{\mu}{\mu_0}R^b_s u_b.
\end{aligned}$$

Proof. For completeness, we include a proof of the well known first statement [9, II2]. Given a vector field X, let $\mathcal{L}_X$ denote Lie derivative. Define a derivation on tensors by $A_X = \mathcal{L}_X - \nabla_X$; on vector fields we have $A_XY = -\nabla_Y X$. Let ϕ_t be a family of diffeomorphisms of M with $\phi_0 = \mathrm{Id}$, $(\mathrm{d}/\mathrm{d}t)|_{t=0}\phi_t = u$. Since the metric is parallel, we get

$$\begin{aligned}
(Lu)_{ab} &= \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\phi_t^* g\right)(\partial_a, \partial_b) = (\mathcal{L}_u g)(\partial_a, \partial_b)\\
&= (A_u g)(\partial_a, \partial_b)\\
&= A_u(g(\partial_a, \partial_b)) - g(A_u\partial_a, \partial_b) - g(\partial_a, A_u\partial_b)\\
&= g(\nabla_a u, \partial_b) + g(\partial_a, \nabla_b u),
\end{aligned}$$

since A_u vanishes on functions. The last line equals

$$(\nabla_a u)^i g_{ib} + (\nabla_b u)^i g_{ia} = (\nabla_a u)_b + (\nabla_b u)_a.$$

For the second equation, we have

$$\begin{aligned}
\langle Lu, \omega\rangle &= \int_M \langle((\nabla_a u)_b + (\nabla_b u)_a)\partial_a \otimes \partial_b, \omega_{cd}\partial_c \otimes \partial_d\rangle\,\mathrm{dvol}\\
&= \int_M ((\nabla_a u)_b + (\nabla_b u)_a)\omega^{ab}\,\mathrm{dvol}
\end{aligned}$$

$$= -2 \int_M u_b \nabla_a \omega^{ab}\,\mathrm{dvol} = -2 \int_M u_c \nabla_a (\omega_b^c g^{ba})\,\mathrm{dvol}$$

$$= -2 \int_M u_c g^{ba} \nabla_a \omega_b^c\,\mathrm{dvol} = -2\langle u, g^{ba}\nabla_a \omega_b^c\rangle.$$

Thus $(L^*\omega)^a = -2\nabla^b \omega_b^a$.

For the third equation, starting as above we get

$$\langle L_u, \omega\rangle_g = \int_M ((\nabla_a u)_b + (\nabla_b u)_a)\omega_{cd} g^{ac} g^{bd}\,\mathrm{d}\mu$$

$$= -2 \int_M u_a g^{cl} \nabla_c \omega_{lb} g^{ab}\,\mathrm{d}\mu = -2 \int_M u^s g^{cl} \nabla_c \omega_{ls}\,\mathrm{d}\mu$$

$$= -2 \int_M u^s (g_0)_{st} (g_0)^{tv} g^{cl} \nabla_c \omega_{lv} \frac{\mu}{\mu_0}\,\mathrm{d}\mu_0$$

$$= \left\langle u, -2(g_0)^{tv} g^{cl} \nabla_c \omega_{lv} \frac{\mu}{\mu_0}\right\rangle.$$

Thus

$$(\overline{L}^*\omega)^a = -2(g_0)^{tv} \frac{\mu}{\mu_0} \nabla^l \omega_{lv}.$$

For the fourth equation, we compute

$$\begin{aligned}
(\Delta u)^a &= -2\nabla^b((Lu)_b^a) = -2\nabla^b(g^{ac} g_{bd} \nabla_c u^d + g^{ac} g_{cd} \nabla_b^d) \\
&= -2\nabla^b(g_{bd}\nabla^a u^d + \nabla_b u^a) = -2(\nabla^b \nabla_b u^a + g_{bd} \nabla^b \nabla^a u^d) \\
&= -2(\nabla^b \nabla_b u^a + g_{bd}(\nabla^a \nabla^b + R^{ab}))u^d \\
&= -2(\nabla^b \nabla_b u^a + \nabla^a \nabla_b u^b + R_b^a u^b).
\end{aligned}$$

We leave the proof of the last statement to the reader. □

The following is a straightforward consequence of Lemma 3.2.

Corollary 3.2. *The symbol of Δ is given by*

$$\sigma(\Delta) = -2(p_0(x,\xi) + p_1(x,\xi) + p_2(x,\xi)),$$

where p_i $(i = 0, 1, 2)$ is homogeneous of degree $2 - i$ in ξ and is given by

$$\begin{aligned}
p_2(x,\xi) &= [|\xi|^2 \delta_j^k + \xi^k \xi_j], \\
p_1(x,\xi) &= i[3g^{lm}\Gamma_{lj}^k \xi_m + g^{lm}\Gamma_{lm}^t \xi_t \delta_j^k + g^{lm}\Gamma_{ml}^k \xi_j + g^{mk}\Gamma_{jm}^l \xi_l], \\
p_0(x,\xi) &= -[g^{lm}\partial_l \Gamma_{mj}^k + g^{kl}\partial_l \Gamma_{mj}^m - g^{ml}\Gamma_{ml}^t \Gamma_{tj}^k + g^{ml}\Gamma_{lt}^k \Gamma_{jm}^t + R_j^k].
\end{aligned}$$

In particular, Δ is elliptic. Thus $\overline{\Delta} = \overline{\Delta}_\alpha$ is elliptic for α close to zero.

Note that the principal symbol of Δ is not scalar as for usual Laplacians.

Proof of Proposition 3.1. By the ellipticity of Δ and [6, Lemma 1.7.4], $\mathrm{Tr}(\mathrm{e}^{-t\Delta})$ has an asymptotic expansion $\sum_k A_k t^{k-(n/2)}$, and hence so does $\mathrm{Tr}(B\mathrm{e}^{-t\Delta})$. Moreover, the coefficients A_k are integrals $\int_M a_k$ of polynomials in the jets of the symbol of Δ. Since these polynomials are independent of coordinates, by a standard argument the a_k must be polynomials of curvature expressions and their covariant derivatives (and constants). In particular, the asymptotic expansion of $\mathrm{Tr}(\mathrm{tr}(N)\mathrm{e}^{-t\Delta})$ is just $\sum_k B_k t^{k-(n/2)}$, where $B_k = \int_M \mathrm{tr}(N)a_k$. For the term $b_{n/2}$, a homogeneity count shows that no constant terms appear. Similarly, for the term A in B given before (3.16), the asymptotic expansion of $\mathrm{Tr}(A\mathrm{e}^{-t\Delta})$ has coefficients which are integrals of expressions involving N and curvature terms. The important point here is that no derivatives of N occur [6, Lemma 1.7.7]. Thus if R denotes a generic curvature term, $b_{n/2}$ will also contain terms of the form $\mathrm{tr}(N)R$, as well as terms given by contracting indices in N against indices in R and then contracting all remaining indices. □

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